A Note on the Sources of Portfolio Returns: Underlying Stock Returns and the Excess Growth Rate

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ABSTRACT

A portfolio's compound return over time is not simply the weighted sum of the compound returns of its underlying stocks. Instead, it is due to (a) the underlying constituent stocks' compound returns, and (b) a component induced by constituent covariances. This can be important. The average smallest-cap decile portfolio outperformed its largest-cap counterpart by 44 basis points per month (bps/mo), but the smallest-cap decile stock constituents on average *underperformed* their largest-cap counterparts by 74 bps/mo. Thus, the "size effect" is not a small-firm effect, but a small-firm *portfolio* effect. In contrast, our high-minus-low (HML) and up-minus-down (UMD) portfolios outperformed because their individual stock constituents outperformed on average. Value and momentum are simultaneously portfolio and individual stock effects.

It is tempting to assume that the performance of a portfolio is equivalent to the performance of its underlying stocks. Unfortunately, this assumption can lead to incorrect inferences regarding the portfolio's constituents and can obscure richer observations about a portfolio's performance and that of its underlying stocks. This paper focuses on two distinct sources of portfolio performance. The first source arises from the familiar first moment of the portfolio's underlying stocks' returns.[1] The second source, though not a second order effect, arises from the variances and covariances of the portfolio's underlying stocks. We empirically estimate these sources based on the Fernholz and Shay (1982) mathematical model of portfolio returns.

The following simple example illustrates the underlying economic issue that is the focus of this paper. Suppose there are four stocks with returns over two periods as presented in Table 1. First, consider stocks A_1 and A_2. Each stock doubles in value one period and loses half its value in the other period. Both stocks will have a 25% average raw return over the two periods, but an average log return of zero. Even though the average log return is zero for each stock, an equal-weighted portfolio (EW_A) of the two stocks earns a 22.31% log return per period.

Now consider stocks B_1 and B_2, which both have raw returns of 25% (log returns of 22.31%) each period. An equal-weighted portfolio (EW_B) of these two stocks will have an arithmetic return of 25% (log return of 22.31%) per period. Thus, portfolio EW_B performs identically to portfolio EW_A, but the sources of the returns are strikingly different. In portfolio EW_A the return is driven by the variance of the underlying stocks. However, the return of portfolio EW_B is a direct result of the log returns of the two stocks. We provide a method for parsing portfolio returns into these two distinct sources.

Our analysis primarily concerns the cross-sectional aggregation of individual stock returns to portfolio returns. This concept is reflected in Table 1 by the fact that a portfolio's log return does not necessarily equal the average log return of its constituent stocks. In the case EW_A, the average

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[1]We adopt the same terminology as Campbell *et al.* (1998) when referring to returns. Unless otherwise indicated, "return" refers to the continuously compounded return on a stock or portfolio. This may be presented as the continuous return, log return, geometric return, compound return, or growth rate in other contexts and captures the average growth rate of a stock or portfolio over multiple time periods. In an effort to be clear, we sometimes emphasize the distinction from simple (i.e., raw or arithmetic) returns with explicit references to compound returns and raw returns.

	Raw Returns				Log Returns			
	Period 1	Period 2	Average	Variance	Period 1	Period 2	Average	Variance
A_1	+100%	−50%	+25%	0.5625	+0.6931	−0.6931	0.0000	0.4805
A_2	−50%	+100%	+25%	0.5625	−0.6931	+0.6931	0.0000	0.4805
EW_A	**+25%**	**+25%**	**+25%**	**0.0000**	**+0.2231**	**+0.2231**	**+0.2231**	**0.0000**
B_1	+25%	+25%	+25%	0.0000	+0.2231	+0.2231	+0.2231	0.0000
B_2	+25%	+25%	+25%	0.0000	+0.2231	+0.2231	+0.2231	0.0000
EW_B	**+25%**	**+25%**	**+25%**	**0.0000**	**+0.2231**	**+0.2231**	**+0.2231**	**0.0000**

Table 1: Examples of the Sources of Returns for Stocks and Portfolios.

log return for the portfolio is 22.31%, while the average log return of its constituent stocks is zero. This phenomenon is distinct from the more widely recognized fact that average log returns for an individual stock over time do not simply equal the log of the average raw returns for that stock over time. For example, the average log return for stock A_1 is zero, but the log of the average raw return of 25% is 22.31% for A_1.

Fernholz and Shay (1982), hereafter FS, offer what we believe to be the first rigorous mathematical analysis to identify the sources of the long-term performance of a portfolio. Specifically, FS show that a portfolio that is rebalanced to the same constant weights has a compound return that is a function of the underlying stocks' compound returns and an "excess growth rate" that is due to the difference between the stocks' variances and the (diversified) portfolio's variance.[2] This dichotomy implies that a portfolio's return can be increased by 1) choosing stocks that have higher returns than their peers; and/or 2) choosing a more favorable mix of stocks based on their variances and covariances. The former component arises from differences among stocks' cross-sectional returns, while the latter component arises from differences among stocks' volatilities and covariances.

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[2]Brennan and Schwartz (1985) independently offer a similar derivation of what they term a positive "bias" in the return of an equal-weighted portfolio (a special case of the FS analysis) over and above the return of an equal-weighted geometric index. Booth and Fama (1992) separately derive a similar decomposition of a portfolio's return and attribute a "diversification return" to an asset's contribution to the portfolio return that is in excess of the asset's compound return. Booth and Fama (1992) provide some analysis at the asset allocation level across indexes, but do not examine the potential effects that occur within portfolios across stocks.

We illustrate the importance of the FS model by empirically estimating the sources of portfolios' returns for several characteristic-based portfolios that appear frequently in the literature. Our results reveal that a portfolio of stocks from the smallest market cap decile have average monthly compound returns of 44 bps per month more than a large stock portfolio, but the average compound returns of stocks in the small-cap portfolio are 74 bps lower than the returns of stocks in the large-cap portfolio. The difference of about 118 bps between these two estimates derives from the excess growth rate that is induced by the higher variance of stocks in the small-cap portfolio. When sorted by book-to-market ratios, the highest book-to-market portfolio has portfolio monthly compound returns that are 127 bps higher than the lowest book-to-market portfolio. In contrast to size portfolios, the difference in portfolio compound returns is driven primarily by the difference in the average underlying stock compound returns for book-to-market portfolios, as the highest book-to-market portfolio's stocks have an average compound return that is 157 bps higher than the average compound return for the stocks in the lowest book-to-market portfolio. Portfolios based on past returns and estimates of beta display both weaker and more non-linear patterns across decile portfolios, yet they also exhibit meaningful differences between portfolio returns and underlying average stock returns.

Our parsing reveals interesting new insights into the sources of portfolio returns in general, and the sources of returns to oft-studied characteristic-based portfolios, such as size (Banz (1981)), market-to-book (Fama and French (1992)), and momentum (Carhart (1997)) portfolios, in particular. For example, the outperformance of small firm portfolios can be attributed to the influence of the constituent stocks' variances on the portfolio's return, not to the returns of the underlying stocks themselves. Our key results illustrate this with the seemingly paradoxical result that *portfolios* of small stocks outperform, while small stocks underperform, on average. The apparent paradox of having outperforming portfolios comprised of underperforming stocks is easily resolved in the mathematical parsing of portfolio returns that we present herein.

1   Background

Numerous academic studies have demonstrated how cross-sectional variation in the characteristics of stocks is related to portfolio performance

over time Brown and Warner (1980), Barber and Lyon (1997), Kothari and Warner (1997), Lyon *et al.* (1999), Boynton and Oppenheimer (2006), and Albuquerque (2012). However, this generalization from individual stocks to portfolios and vice versa is subject to inherent limitations as the time-series patterns of individual stock returns do not directly aggregate to portfolio return patterns. We therefore approach the issue from the opposite direction. By imposing precise and consistent measurement methods on stock and portfolio returns by using continuously compounded (i.e., log) returns and fixed rebalancing frequencies, we are able to parse portfolio returns into the underlying contributions made by the constituent stocks. This allows us to undertake an analysis of the cross-sectional characteristics of individual stocks and their contribution to a portfolio's return that was not utilized in past studies.

Firm size is a good example of a cross-sectional stock-level characteristic that can be used to show how economic explanations can be greatly refined through our decomposition of portfolio returns. Discussions of the size effect typically fall into two categories: 1) portfolio comparisons: "Portfolios of small firms outperform portfolios of large firms"; and 2) stock comparisons: "Small firms earn higher returns than large firms." For example, Bodie *et al.* (2011) fall into the former case, noting that "Average annual returns between 1926 and 2006 are consistently higher on small-firm portfolios" (p. 361). However, others often provide the latter explanation. For example, Reilly and Brown (2006) note in reference to studies of the size effect, "During most periods they found the negative relationship between size and return; but, during others (such as 1967 to 1975), they found that large firms outperformed the small firms" (p. 180). Similarly, Sharpe (2008) notes "small stocks seem to have performed better than large stocks"; and Malkiel (1999) describes the size effect as, "the tendency over long periods of time for small stocks to do better than large stocks" (p. 250). An often-overlooked problem here is that, with compound returns, one category does not imply the other. Indeed, it is possible to have a portfolio of stocks that outperform over the long term, even though the underlying stocks are below-average performers.

To decompose a portfolio's return, we begin with the returns of the portfolio's underlying assets. Let $\gamma_{i,t}(h)$ be the compound return (i.e., continuously compounded or log return) for stock i at time t over a holding period of length h. If $\gamma_{i,t}(h)$ is normally distributed (i.e., raw stock returns are log-normally distributed) with mean $\bar{\gamma}_i(h)$ and variance $\sigma_i^2(h)$, then

the average compound return and the average holding period return, $\bar{r}_i(h)$, are related to one another as[3]

$$\bar{r}_i(h) \approx \bar{\gamma}_i(h) + \frac{\sigma_i^2(h)}{2}. \tag{1}$$

If stock holding period returns are not log-normal then it is an empirical question whether (1) is a useful approximation. If stock returns are log-normally distributed, then portfolio returns are not log-normally distributed. Similarly, if portfolio returns are log-normally distributed, the average holding period return of portfolio p can be expressed analogously as for individual stocks:

$$\bar{r}_p(h) \approx \bar{\gamma}_p(h) + \frac{\sigma_p^2(h)}{2}. \tag{2}$$

Using the definition of the arithmetic average return on a portfolio and Eq. (1), the arithmetic average return on a portfolio that maintains a constant weight in each stock is

$$\bar{r}_p(h) = \sum_{i=1}^{N} w_i \cdot \bar{r}_i(h) \\ = \sum_{i=1}^{N} w_i \cdot \bar{\gamma}_i(h) + \frac{1}{2} \cdot \sum_{i=1}^{N} w_i \cdot \sigma_i^2(h), \tag{3}$$

where w_i is the portfolio weight in stock i. Note that this portfolio is implicitly rebalanced every holding period h to maintain the constant weights. Combining (3) and (1), the average compound return on a portfolio is

$$\bar{\gamma}_p(h) = \sum_{i=1}^{N} w_i \cdot \bar{\gamma}_i(h) + \frac{1}{2} \cdot \left[\sum_{i=1}^{N} w_i \cdot \sigma_i^2(h) - \sigma_p^2(h) \right]. \tag{4}$$

Using continuous rebalancing, Fernholz and Shay (1982) derive a continuous-time version of (4) and identify the second term as a portfolio's *excess growth rate*. We denote the discrete time portfolio excess growth rate as

$$\gamma_p^*(h) = \frac{1}{2} \cdot \left[\sum_{i=1}^{N} w_i \cdot \sigma_i^2(h) - \sigma_p^2(h) \right]. \tag{5}$$

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[3]The exact relationship is $\ln[1 + \bar{r}_i(h)] = \bar{\gamma}_i(h) + \sigma_i^2(h)/2$. We use the approximation formula for ease of exposition and illustration, ignoring the approximation beyond Eq. (3) and assuming it holds exactly. The empirical validity of this assumption and approximation are confirmed by the results herein.

The purpose in identifying the holding period length h in the discrete time version is to emphasize the importance of matching the measurement interval of the variances and covariances with the frequency of rebalancing to the constant weights. For example, when using monthly returns to estimate Eq. (4), monthly rebalancing is implied.

Equation (4) shows that a portfolio's compound return comes from two primary sources: 1) the weighted average compound returns of the constituent stocks; and 2) the portfolio's excess growth rate. Not surprisingly, weighting stocks with high returns benefits a portfolio's return. Less obvious is the impact to a portfolio's return that arises from the excess growth rate. Specifically, any portfolio that heavily weights stocks with relatively high variances should benefit, assuming that the stocks are not perfectly correlated so that the portfolio's variance is not increased proportionally.[4] Potential candidates for portfolios that put relatively high weight on high-variance stocks are equal-weighted portfolios, contrarian portfolios, or small stock portfolios. To the extent that the stocks in these portfolios have higher variances, they should have higher excess growth rates.

Equation (4) is useful beyond its implications for portfolio analysis. Specifically, many studies utilize, either explicitly or implicitly, constant-weight portfolios in their analysis. Indeed, any study employing panel data that averages the returns across stocks and then considers the average of this measure over time is subject to these effects that act on a portfolio's return. The return of Portfolio A could exceed the return of Portfolio B if: (1) the stocks in Portfolio A have higher returns than the stocks in portfolio B; *or* (2) the excess growth rate of Portfolio A exceeds that of Portfolio B; *or* both. It would be inadvisable to infer from a higher return of Portfolio A compared to Portfolio B that the stocks in Portfolio A have higher returns compared to stocks in Portfolio B just by considering the portfolios' returns. Equation (4) allows researchers to parse the effects and distinguish between an interpretation that implies something about the *underlying firms* and another interpretation that implies something only about *portfolios* of such firms. In some circumstances, this subtlety might

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[4]Booth and Fama (1992) derive an analogous expression for a portfolio's average log return from the Taylor series expansion of the natural logarithm, referring to the excess growth effect as the "diversification return." Their analysis focuses on the diversification return to various asset-class-level indexes and does not consider the effect within the indexes and asset classes that they consider.

be considered negligible semantics, while in other circumstances it might be critical to the interpretation and implications of the study.

2 The Excess Growth Rates of Factor-based Portfolios

Fama and French (1992) (hereafter, FF), among others, show a "size" effect in which portfolios of small (low market equity (ME)) stocks have higher average returns compared to portfolios of large (high ME) stocks, as well as a book-to-market effect where portfolios of high book-to-market (high BE/ME) stocks have higher average returns compared to portfolios of low book-to-market (low BE/ME) stocks. Carhart (1997) incorporates the momentum effect of Jegadeesh and Titman (1993) into FF's model and shows that past performance can explain cross-sectional differences in the portfolios of mutual funds. While the momentum effect implies that current performance is positively related to (short-term) past performance, earlier work by De Bondt and Thaler (1987) suggests an opposite pattern of "reversals" over longer horizons.

   The analysis below shows that the size and reversal effects are driven by the relative excess growth rates of portfolios sorted by size or long-term past performance. In contrast, the book-to-market and short-term momentum effects are actual characteristics of the underlying stock returns. Specifically, small stock and long-term reversal portfolios have higher returns because small stocks and stocks that have performed poorly over the previous 3–5 year period have higher variances, on average, leading to higher excess growth rates for portfolios composed of these stocks. These portfolios have higher returns despite the fact that small and contrarian stocks have lower compound returns, on average. In other words, these stocks do not outperform on average, but portfolios composed of them do. In contrast, high book-to-market portfolios and short-term momentum portfolios outperform because the constituent stocks have higher average compound returns.

   For the sample period of January 1960 through December 2012, we match monthly raw return and capitalization (ME) data for non-financial common stocks from the Center for Research in Security Prices (CRSP) with book value of equity (BE) data in Compustat. We construct size, book-to-market and beta decile portfolios in a similar manner as in FF[5]. As in FF, the

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   [5]We find that using calendar-year returns gives almost identical results to our analysis as

ME and beta decile breakpoints are determined using only New York Stock Exchange (NYSE) stocks, while stocks from NYSE, the American Stock Exchange (AMEX), and Nasdaq are used to determine decile breakpoints for book-to-market portfolios. To analyze momentum and reversal effects, we sort stocks based on their returns in period $t-1$. We explore a wide range of prior return horizons, ranging from 1 to 60 months, in 3-month intervals, but we limit our reported results to portfolios formed based on prior 6-month and 5-year periods.

Once our characteristic-based portfolios are formed, equal-weighted average portfolio compound returns are calculated for each month of year t.[6] As in the previous section, the natural log of (one plus) a stock's raw return is used to calculate the stock's compound return. Similarly, a month's actual portfolio compound return is the natural log of (one plus) the portfolio's raw return in that month. Annual variances are calculated using monthly log returns.

2.1 Size Decile Portfolios

Panel A of Table 2 shows the average raw returns, compound returns, and the estimated components of returns from Eq. (4) for the size (ME) decile portfolios. The first column of estimates contains the arithmetic average portfolio raw return, $\bar{r}_p$, over the sample period. The pattern of arithmetic average raw returns across size deciles is similar to that in Fama and French (1992).[7] The average return of the small stock (Low ME) portfolios is generally higher than the average return of the large stock (High ME) portfolios. While widely known, it is important to note that the true growth in an investor's wealth as given by the average actual compound return, $\bar{\gamma}_p$(act.), when holding one of these portfolios is not reflected accurately by the arithmetic average raw return of the portfolio's constituent stocks. While also possibly known, but almost never explicitly acknowledged, the average portfolio compound returns, $\bar{\gamma}_p$(act.), are poor

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using July-to-June returns and we therefore report results based on calendar-year returns.

[6]Although our reported results focus on average monthly returns over year t, we also examine, but do not report, results over both longer and shorter performance evaluation periods. We find that our portfolio return decomposition continues to provide insights into the sources of portfolio returns over these alternative periods, although a detailed analysis of this would be beyond the scope of the current study.

[7]We have also conducted our analysis using the FF sample period of 1963 through 1990. We obtain results that are nearly identical to those for our 53-year sample period.

estimates of the average compound returns of the portfolio's underlying assets, $\sum_{i=1}^{N} w_i \cdot \bar{\gamma}_i$.

The pattern in average portfolio compound returns, $\bar{\gamma}_p(\text{act.})$, is similar to the pattern in average portfolio raw returns, $\bar{r}_p$. Likewise, the pattern in estimated average portfolio compound returns, $\bar{\gamma}_p(\text{est.})$, which is based on constituent stocks' average compound returns and variances and portfolio variance in Eq. (4), conforms quite well to the pattern in actual portfolio compound returns. However, the average compound return to the underlying assets, $\sum_{i=1}^{N} w_i \cdot \bar{\gamma}_i$, reveals a pattern that is strikingly different than that of the average portfolio returns. Specifically, smaller stocks have lower average compound returns than larger stocks. The excess growth rate reconciles this apparent contradiction. The decomposition using Eq. (4) and Eq. (5) shows that the outperformance of the portfolios of smaller stocks is driven by the excess growth rate, $\bar{\gamma}_p^*(\text{est.})$, of those portfolios.

The average portfolio excess growth rates decrease almost monotonically as the size decile increases from the smallest to the largest stocks. The same pattern appears in the estimated portfolio return that is calculated from the components of Eq. (4). The estimated average portfolio return, $\bar{\gamma}_p(\text{est.})$, in the third column of Table 2 corresponds well to the actual average portfolio return, $\bar{\gamma}_p(\text{act.})$, in the second column.[8]

The decrease in portfolio returns as the size decile increases occurs despite the fact that the underlying average stock compound returns increase as the size decile increases. Thus, the driving factor in the outperformance of small stock portfolios is the variance of the underlying stocks. For the entire sample period, the smallest size decile portfolio holds stocks with an average variance of 0.0308 per month, while the largest size decile portfolio holds stocks with an average variance of only 0.0067 per month. The variance of the small stocks is enough to generate portfolio excess growth rates that compensate for the small stocks' lower returns. The decline in the portfolio returns as the size of the underlying stocks increases is driven by the decline in the portfolios' excess growth rates due to the decline in the underlying stocks' variances.

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[8]We have also validated, but do not report, the empirical decomposition of Eq. (4) by estimating the portfolio return from the equation's components and comparing it to the actual portfolio return each year. A regression of the yearly estimated returns on the returns for an equal-weighted portfolio of CRSP stocks yields a slope coefficient estimate of 1.0, a statistically insignificant intercept coefficient, and an R-squared exceeding 0.999.

Portfolio	$\bar{r}_p$	$\bar{\gamma}_p$(act.)	$\bar{\gamma}_p$(est.)	$\sum_{i=1}^{N} w_i \cdot \bar{\gamma}_i$	$\sum_{i=1}^{N} w_i \cdot \sigma_i^2$	σ_p^2	$\bar{\gamma}_p^*$(est.)
Panel A: Size Deciles							
Low	0.0148	0.0121	0.0115	−0.0016	0.0308	0.0047	0.0130
2	0.0113	0.0090	0.0095	0.0020	0.0195	0.0045	0.0075
3	0.0121	0.0099	0.0118	0.0056	0.0166	0.0043	0.0062
4	0.0116	0.0096	0.0106	0.0051	0.0149	0.0040	0.0055
5	0.0117	0.0098	0.0111	0.0063	0.0132	0.0036	0.0048
6	0.0112	0.0095	0.0101	0.0060	0.0115	0.0032	0.0041
7	0.0114	0.0098	0.0110	0.0074	0.0102	0.0030	0.0036
8	0.0106	0.0091	0.0100	0.0066	0.0095	0.0028	0.0033
9	0.0104	0.0091	0.0096	0.0067	0.0083	0.0024	0.0029
High	0.0088	0.0077	0.0082	0.0058	0.0067	0.0020	0.0024
Panel B: Book-to-Market Deciles							
Low	0.0086	0.0055	0.0050	−0.0086	0.0335	0.0062	0.0136
2	0.0089	0.0065	0.0065	−0.0028	0.0235	0.0048	0.0094
3	0.0098	0.0076	0.0076	−0.0005	0.0204	0.0041	0.0081
4	0.0093	0.0074	0.0074	0.0000	0.0185	0.0037	0.0074
5	0.0115	0.0097	0.0099	0.0032	0.0170	0.0035	0.0067
6	0.0123	0.0105	0.0106	0.0040	0.0164	0.0033	0.0066
7	0.0135	0.0118	0.0118	0.0055	0.0157	0.0031	0.0063
8	0.0148	0.0131	0.0133	0.0066	0.0163	0.0030	0.0066
9	0.0165	0.0146	0.0145	0.0071	0.0183	0.0035	0.0074
High	0.0213	0.0184	0.0185	0.0071	0.0276	0.0049	0.0114
Panel C: Beta Deciles							
Low	0.0104	0.0103	0.0102	0.0030	0.0152	0.0008	0.0072
2	0.0102	0.0101	0.0102	0.0048	0.0117	0.0010	0.0053
3	0.0112	0.0110	0.0110	0.0055	0.0124	0.0015	0.0055
4	0.0107	0.0106	0.0107	0.0047	0.0137	0.0019	0.0059
5	0.0106	0.0104	0.0105	0.0041	0.0152	0.0023	0.0065
6	0.0112	0.0109	0.0110	0.0039	0.0170	0.0028	0.0071
7	0.0107	0.0104	0.0104	0.0027	0.0191	0.0035	0.0078
8	0.0105	0.0101	0.0103	0.0012	0.0227	0.0044	0.0091
9	0.0103	0.0099	0.0099	−0.0007	0.0272	0.0059	0.0107
High	0.0113	0.0104	0.0102	−0.0054	0.0424	0.0112	0.0156

Table 2: Size and Book-to-Market Decile Portfolios.

Portfolio	$\bar{r}_p$	$\bar{\gamma}_p$(act.)	$\bar{\gamma}_p$(est.)	$\sum_{i=1}^{N} w_i \cdot \bar{\gamma}_i$	$\sum_{i=1}^{N} w_i \cdot \sigma_i^2$	σ_p^2	$\bar{\gamma}_p^*$(est.)
Panel D: Prior Short-term (6-month) Performance Deciles							
Low	0.0092	0.0086	0.0092	−0.0131	0.0522	0.0076	0.0223
2	0.0082	0.0078	0.0078	−0.0045	0.0292	0.0047	0.0123
3	0.0089	0.0086	0.0086	−0.0003	0.0214	0.0035	0.0089
4	0.0094	0.0092	0.0091	0.0020	0.0171	0.0028	0.0071
5	0.0101	0.0099	0.0098	0.0037	0.0145	0.0024	0.0061
6	0.0106	0.0104	0.0104	0.0049	0.0131	0.0021	0.0055
7	0.0113	0.0111	0.0110	0.0054	0.0133	0.0021	0.0056
8	0.0109	0.0107	0.0107	0.0048	0.0142	0.0024	0.0059
9	0.0107	0.0105	0.0104	0.0034	0.0172	0.0031	0.0070
High	0.0094	0.0090	0.0090	−0.0021	0.0271	0.0049	0.0111
Panel E: Prior Long-term (5-year) Performance Deciles							
Low	0.0162	0.0155	0.0161	−0.0078	0.0559	0.0081	0.0239
2	0.0103	0.0099	0.0098	−0.0044	0.0333	0.0048	0.0142
3	0.0095	0.0092	0.0092	−0.0011	0.0242	0.0037	0.0103
4	0.0091	0.0089	0.0089	0.0010	0.0187	0.0029	0.0079
5	0.0094	0.0092	0.0091	0.0027	0.0152	0.0023	0.0064
6	0.0101	0.0099	0.0099	0.0041	0.0135	0.0021	0.0057
7	0.0094	0.0092	0.0092	0.0039	0.0127	0.0022	0.0053
8	0.0098	0.0096	0.0096	0.0043	0.0128	0.0023	0.0053
9	0.0090	0.0088	0.0088	0.0029	0.0144	0.0027	0.0059
High	0.0065	0.0063	0.0063	−0.0015	0.0197	0.0042	0.0078

Table 2: *(Continued)*

Description: This table reports the summary statistics for the estimates of the terms of Eq. (4) for equal-weighted portfolios formed by size (market capitalization) book-to-market, beta, and past performance deciles for the sample period of January 1960 through December 2012. The terms for the weighted average return and the weighted average stock variance ($\sum_{i=1}^{N} w_i \cdot \bar{\gamma}_i$ and $\sum_{i=1}^{N} w_i \cdot \sigma_i^2$, respectively) are estimated each year using monthly data, with that year's estimate applied to each month within that year. The portfolio variance, σ_p^2, reflects the variance across all months in the sample period, while all other columns reflect the average of the given parameter across all months in the sample period. The portfolio's arithmetic average raw return, $\bar{r}_p$, is reported in the first column of estimates. The term $\bar{\gamma}_p$(act.) reports the portfolio's average actual compound return (i.e., log return), while $\bar{\gamma}_p(est.)$ is the portfolio's estimated average compound return (i.e., log return) using Eq. (4). The remaining terms are defined in Eq. (4).

Interpretation: Portfolio compound returns can differ from the underlying components of those returns for portfolios formed from several stock characteristics. In Panel A, portfolio returns ($\bar{\gamma}_p$(act.)) are negatively associated with market-cap even though average underlying stock returns ($\sum_{i=1}^{N} w_i \cdot \bar{\gamma}_i$) are positively related to market cap. In Panel B, both portfolio returns and average stock returns are positively associated with book-to-market ratios. In Panel C, results are consistent with both portfolios and average stock returns having only weak and non-linear associations with estimates of beta. In Panel D, short-term past returns have a positive but non-linear association with both portfolio returns and average stock returns. In Panel E, long-term past returns are positively associated with average stock returns but negatively associated with portfolio returns.

2.2   Book-to-Market Portfolios

Panel B of Table 2 shows the average returns and the estimated components from Eq. (4) for portfolios formed using book-to-market ratio (BE/ME) deciles. Again, these results are nearly identical to those of FF, in that the lowest deciles have the lowest average performance. In contrast to the results for the size decile portfolios, the pattern of average portfolio returns across book-to-market deciles appears to be driven primarily by the underlying stocks' average compound returns. That is, the average returns of low BE/ME stocks are lower than the returns of high BE/ME stocks.

The average stock returns increase monotonically from the lowest BE/ME decile to the highest BE/ME decile. The spread between the highest and lowest BE/ME deciles' average stock returns is 1.57%, while the spread between the highest and lowest size deciles is only 0.74% (and in the opposite direction as expected, in that large stocks have higher average compound returns than small stocks).

As in the size decile results, the average portfolio excess growth rates decrease monotonically as the BE/ME decile increases because the variances of the portfolios' underlying stocks decrease monotonically, except for the highest BE/ME decile. Across the entire sample period, the lowest BE/ME decile portfolio's excess growth rate is 1.36% compared to the highest BE/ME decile portfolio's excess growth rate of only 1.14%. However, this decrease is smaller in magnitude than in the size deciles and is more than offset by the increase in the underlying stocks' average returns. Unlike size decile portfolios, the driving factor in the outperformance of low BE/ME stock portfolios is the average compound returns of the underlying stocks.

2.3   Beta Portfolios

When stocks are sorted by their (lagged) beta estimates, we find no relationship between beta decile rankings and portfolio returns. This is consistent with FF. Panel C of Table 2 presents the components of returns based on deciles formed from beta, as estimated over the previous 5-year period using the CRSP value-weighted index. Although portfolio returns are unrelated to beta, average stock returns show a non-linear and negative association with beta. Contrary to the traditional Capital Asset Pricing Model (CAPM), we find that higher beta estimates are generally associated with lower average stock returns. The negative association between average stock returns and beta is almost perfectly balanced by a positive

association between excess growth rates and beta, resulting in essentially no correlation between beta and portfolio returns.

2.4 *Portfolios Formed Based on Past Performance*

Existing studies of the persistence of past performance are complicated by several considerations. Returns at very short horizons (i.e., less than one month) may display substantial microstructure effects and other biases. At short-to-medium horizons, a momentum effect has been documented where top-performing portfolios in period $t-1$ have above-average performance in period t. At longer horizons (i.e., 3 to 5 years) a reversal, or contrarian, effect appears to dominate, with portfolios of the worst performing stocks from period $t-1$ performing relatively better in period t. Table 2 displays the components of monthly returns over year t for portfolios formed from performance during prior horizons of six months (Panel D) and five years (Panel E).

We find that the momentum effect is a robust empirical characteristic of average stock returns ($\sum_{i=1}^{N} w_i \cdot \bar{\gamma}_i$) in year t, regardless of the prior return horizon used to form portfolios. That is, if a portfolio's stocks had high returns in the prior period then that portfolio's stocks tend to have high returns, on average, in the current year. Although the relationship between average stock returns and prior performance is generally upward sloping, the pattern is non-linear. The pattern in average compound stock returns increases monotonically at first, peaking at decile 7, and then decreases slightly for the remaining deciles.

The momentum effect that we find for portfolios formed from past 6-month performance periods (consistent with existing papers such as Jegadeesh and Titman (1993)) is also present for portfolios formed from long-term (5 year) past performance periods (apparently contrary to the reversal effect documented by De Bondt and Thaler (1985) and De Bondt and Thaler (1987)). In untabulated tests, we find this result to be robust to a wide range of portfolio formation horizons ranging from one month to five years. That is, a momentum effect in underlying average stock returns (over a current period of one year) is a pervasive effect across all portfolio formation horizons. One important distinction here is that the results mentioned above apply to the average compound returns ($\sum_{i=1}^{N} w_i \cdot \bar{\gamma}_i$) of the stocks in each portfolio, while existing research generally examines portfolio returns ($\bar{r}_p$ or $\bar{\gamma}_p$).

Consistent with the published literature, at the portfolio level we also observe momentum effects for portfolio returns ($\bar{r}_p$) only when portfolios are composed based on past return horizons of two years or less. When portfolios are composed based on prior performance periods of longer than two years, then the excess growth effect dominates over the momentum effect present in the returns ($\sum_{i=1}^{N} w_i \cdot \bar{\gamma}_i$) of the underlying stocks for the stocks that performed poorly in period $t-1$ and we therefore observe a reversal pattern in portfolio returns. More precisely, the portfolios composed of stocks with low returns in the prior 2-to-5 year period have sufficiently large excess growth rates ($\bar{\gamma}_p^*$) arising from the variance of those stocks ($\sum_{i=1}^{N} w_i \cdot \sigma_i^2$) so that the current portfolio return ($\bar{\gamma}_p$) is high enough to display a "reversal" effect, even though the average stock returns ($\sum_{i=1}^{N} w_i \cdot \bar{\gamma}_i$) of the portfolio's constituents are still relatively low on average.

Overall, we find that the momentum effect can be generalized to underlying stock returns only when portfolios are formed based on prior return horizons of two years or less. The contrarian, or reversal, effect for portfolios formed based on long-term (two years or greater) prior performance is only a portfolio-level effect that cannot be generalized to the underlying stocks.

3 The Cross-Section of Stock Returns

Beginning with Fama and MacBeth (1973), cross-sectional regressions have been used to examine the association between stock returns and stock characteristics. Cross-sectional analysis appears to have the desirable property of ease of interpretation, in that the slope coefficients from cross-sectional regressions reflect a direct association between a firm's return (dependent variable) and the firm's characteristics (independent variables). The attribution of the portfolio's return in the previous sections shows that a difference can exist between the returns to a portfolio that is formed on a specific stock characteristic and the average return on the underlying stocks with that characteristic. To further examine the role of stocks' variances in explaining the association between a stock's return and its other characteristics, we estimate Fama-MacBeth-style cross-sectional regressions for both raw and compound (log) stock returns.

The results in the previous section raise questions about the commonly accepted view that smaller firms have higher returns. To further examine

this, we include the same-year variance of a stock's monthly returns (Variance) as an independent variable in cross-sectional regressions, in addition to using ln(ME) and ln(BE/ME) to measure a firm's size and market-to-book ratio, respectively. We include a stock's beta estimate, using the pre-ranking beta estimate from FF, as a control variable. If the size effect in portfolio returns is due to stock variances, then ln(ME) should not be negatively associated with returns when Variance is included in the regression. More importantly, based on the result in the previous section that average stock returns are higher for firms in the larger size deciles, we expect the average slope on ln(ME) to be positive when using log returns as the dependent variable.

We report the time-series average slope coefficients from monthly cross-sectional regressions in Table 3. As shown in Panel A, the average slope on ln(ME) in the monthly raw return univariate regressions is negative over the period from January 1960 through December 2012.[9] The average coefficient on ln(BE/ME) is statistically significant and positive. Variance also takes a significant and positive coefficient estimate, while beta is unrelated to raw returns. In the multivariate regressions with all four explanatory variables, the average coefficients on ln(ME) and beta are insignificant, while the average coefficients on ln(BE/ME) and Variance remain positive and statistically significant.

Our return decomposition and cross-sectional regressions show that the volatility of small stocks is the primary contributor to small stock portfolios' returns, suggesting that caution should be exercised when inferring any long-term return premium on a stock due to ln(ME). With the average cross-sectional correlation between ln(ME) and Variance of log returns being -0.46, the effect of multicollinearity in the cross-sectional regressions must be considered. We note that the average slope coefficient on Variance remains relatively stable when ln(ME) is added to the model. Moreover, the results using stocks' log returns as the dependent variable in regressions casts more doubt on a size effect in individual stock returns. Over the entire sample period, the coefficient on ln(ME) in monthly stock log return regressions is, on average, *positive* and statistically significant. The average coefficient remains positive but statistically insignificant in multivariate regressions. The average coefficient for ln(BE/ME) remains

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[9] We have estimated the cross-sectional regression analysis using the FF sample period of 1963–1990 and obtain similar results.

Panel A: Average Slope Coefficients

	Dependent Variable: Raw Returns				Dependent Variable: Log Returns		
ln(ME)	ln(BE/ME)	Variance	Beta	ln(ME)	ln(BE/ME)	Variance	Beta
−0.0014 (−3.20)				0.0009 (2.28)			
	0.0040 (4.98)				0.0049 (6.42)		
		0.3597 (3.72)				−0.1633 (−1.81)	
			0.0022 (1.33)				−0.0038 (−2.32)
−0.0002 (−0.06)	0.0026 (3.92)	0.4413 (5.03)	−0.0012 (−0.90)	0.0001 (0.26)	0.0025 (3.90)	−0.0708 (−0.87)	−0.0014 (−1.06)

Panel B: *J Test* of ln(ME) and Variance

Dependent Variable: Raw Returns

ln(ME)	$\hat{r}_{(Variance)}$	Variance	$\hat{r}_{(\ln(ME))}$
0.0031 (0.12)	1.8716 (22.79)		
		0.2950 (3.77)	0.8108 (2.07)

Table 3: Average Slope Coefficients from Cross-Sectional Regressions.

Description: In Panel A. each row reports the time-series average slope coefficients from cross-sectional regressions for the sample period of January 1960 through December 2012. Monthly stock raw returns and compound returns (i.e., log returns) are used as dependent variables, with the beginning-of-year market cap (ln(ME)), beginning-of-year book-to-market ratio (ln(BE/ME)), prior 5-year beta estimate, and same-year log return variance (Variance) used as independent variables. Rows one through four each report results from univariate regression models, while row five reports results from a multivariate regression model that includes all four independent variables. Panel B reports the second stage results from a *J Test* (Davidson and MacKinnon, 1981) of ln(ME) and Variance in the cross-sectional regressions. In the first stage, returns are regressed on ln(ME) or Variance. In the second stage, the predicted values of returns ($\hat{r}$) are used along with the actual values of ln(ME) (row 1) or Variance (row 2) to explain actual returns. T-statistics are reported in parentheses below the coefficient estimates.

Interpretation: Panel A of Table 2 compares the usefulness of stocks' market caps (ln(ME)), book-to-market ratios (ln(BE/ME)), variances, and betas, to explain the cross-section of both raw returns and compound (log) returns. The multivariate regression coefficients reported in Row 4 show that variance, but not market cap, is significant in explaining raw returns, although variance is insignificant in explaining log returns. This is consistent with a mechanical relationship between variance and raw returns. The J-test of Panel B confirms that stock variance explains the cross-section of stock returns better than market cap.

positive and significant, while the average coefficient estimate for beta is insignificantly negative in multivariate regressions. The average slope on stocks' own variances is statistically insignificant and negative in explaining log returns.

We conduct a *J* test Davidson and MacKinnon (1981) to determine whether size or variance appears to be the true underlying factor in the cross-section of monthly raw stock returns. In the first stage of this test, returns are regressed on variance (log size) and the resulting coefficient estimates are used to compute a predicted value of returns based on information from variance (size). In the first row of Panel B, actual returns are regressed on the actual values of size and the predicted values of returns when variance is used to forecast returns. The second row of Panel B reports estimates from a model using the actual values of variance and the size-predicted values of returns. The first row indicates that size has no additional power to explain returns in a model that accounts for predicted returns based on information derived from variance. The second row indicates that variance retains the power to explain returns after information from size is incorporated into the model. Therefore, Panel B of Table 3 shows that we are able to reject ln(ME) in favor of Variance as the true underlying factor in explaining the cross-section of stock returns. The results from the *J* test are consistent with the results from the portfolio return decomposition that attributes the source of outperformance in small-stock portfolios relative to large stock portfolios as originating from the variance of the underlying stocks.

Finally, we note that the average slope coefficient for own-stock variance in raw return regressions is close to one-half, as implied by Eq. (1). In also considering the result of no statistically significant relationship between log returns and either ln(ME) or Variance, we conclude that the relationship between raw returns and stock variances simply reflects the mathematical description of stock returns in Eq. (1). That is, the use of arithmetic averages and raw returns rather than compound returns results in apparent relationships that have a *mathematical* rather than an *economic* explanation. When variables that are highly correlated with variance, such as size, are used in research methods that implicitly involve portfolios through the use of arithmetic averages, results must be interpreted with caution and the mathematical impact of variance must be considered.

4 Summary and Conclusion

We employ a mathematical parsing of portfolio compound returns into two key sources: 1) the average compound return of the portfolio's underlying stocks; and 2) the "excess growth rate" of the portfolio that is due to the underlying stocks' return variances and covariances. While the decomposition that we examine has been known to the literature for at least 25 years, it has been rarely cited; and, to our knowledge, we are the first to utilize this mathematical relationship in comparing individual stock characteristics to portfolio returns. We demonstrate that using the return components can yield new and meaningful insights into the sources of performance of commonly examined portfolios in the literature.

We provide new evidence regarding the oft-cited size, book-to-market, and momentum effects by examining the components of the returns to portfolios formed based on these factors.[10] We show that small firms' average compound returns are lower than large firms' average compound returns. However, because small firms generally have higher return variances, portfolios of small stocks have higher excess growth rates than portfolios of large firms. It is the variance-induced excess growth rate that is primarily responsible for the outperformance of small stock portfolios relative to large

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[10]In addition to sorting stocks based on underlying characteristics such as size, book-to-market ratios, and past returns, we also examine portfolios based on stocks' exposures to factors derived from such underlying characteristics. For example, to analyze the size effect, instead of sorting stocks directly based on market cap, we can first run a time-series regression of a stock's returns on the small-minus-big (SMB) portfolio returns and then sort by the coefficient estimate (i.e., factor loading) from this regression.

The factor-based portfolio returns that we observe are consistent with other studies of factor-based portfolios, such as Daniel and Titman (1997). Our portfolio return decomposition shows that the components of portfolio returns display similar patterns regardless of whether portfolios are formed directly from stock characteristics or indirectly based on factor exposures. We therefore do not tabulate results for portfolios formed on the basis of factor loadings.

Our method can also be used to examine additional factor-based portfolios, such as those derived from the liquidity measures of Pastor and Stambaugh (2003). For example, for portfolios formed from exposure to the level of the Pastor-Stambaugh liquidity measure, we find that portfolio returns are positively related to factor loadings across deciles, while the average underlying stock returns display a weak negative (and non-linear) association with the liquidity factor loadings. Therefore, portfolios formed based on exposure to this market-liquidity factor appear to be another example of a portfolio return pattern that is driven by the variance and excess growth rates of the underlying stocks, rather than by the average compound returns of the underlying stocks.

stock portfolios. A similar effect is shown to exist for "contrarian" portfolios. In contrast, the performance of portfolios formed by sorting stocks by their book-to-market ratio is primarily determined by the performance of the underlying stocks. That is, the higher average compound returns of higher book-to-market stocks results in higher portfolio returns, overcoming the offsetting effect that excess growth rates are lower for these portfolios. Stocks displaying short-term momentum (based on 1-month to 2-year prior returns) also display patterns in portfolio returns that are generalizable to the underlying stocks in the portfolio.

Cross-sectional regressions of monthly raw returns and log returns confirm the results from the decomposition of portfolio returns. Specifically, the apparent cross-sectional relationship between size and stock raw returns is not robust to the inclusion of stock variance in the model, while the book-to-market effect is unchanged when variance is included. Furthermore, no cross-sectional relationship is evident between stock log returns and either size or own-return variance, but a firm's book-to-market ratio is positively associated with its average log return.

The implications and applications of decomposing portfolio returns into effects due to stock returns and portfolio excess growth rates are wide-ranging. Whenever the returns of portfolios composed of high-volatility stocks are compared to benchmarks or control portfolios of lower-volatility stocks, we can expect to observe differences due to the excess growth rates of the portfolios. This potentially applies to any portfolio composed of relatively more volatile sample stocks, with the volatility arising due to measurement techniques (i.e., bid-ask bounce), the time period (i.e., January returns), firm characteristics (i.e. small size, high leverage, recent initial public offering (IPO) activity, or low liquidity), or return patterns (i.e., momentum or contrarian). Even in the wider context of other social sciences that study phenomena subject to compound growth rates, any time the growth rate of a high-volatility treatment group is compared to a lower-volatility control group, there could be differences due to the excess growth effect. The current study demonstrates how the concept of an excess growth rate can explain portfolio returns in the context of several common characteristic-based portfolios. The ability of portfolio excess growth rates to explain other documented patterns in portfolio returns is a potentially valuable avenue for future research.

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